NONSMOOTHABLE GROUP ACTIONS ON SPIN 4-MANIFOLDS

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ABSTRACT. We show that every closed, simply connected, spin topological 4-manifold except S^4 and $S^2 \times S^2$ admits a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ for any sufficiently large prime number p which is nonsmoothable for any possible smooth structure.

1. INTRODUCTION

In this article, we call a locally linear action of a group on a topological manifold *nonsmoothable* if the action is not smooth with respect to any possible smooth structure. Several authors have been investigated examples of nonsmoothable group actions on 4-manifolds [11, 10, 8, 1, 14].

We restrict our attention to actions of the cyclic groups of odd prime order which are homologically trivial and pseudofree. A. L. Edmonds constructed such actions on all simply connected 4-manifolds [5]. The main purpose of this article is to show that there is a family of locally linear actions constructed by Edmonds's method which are nonsmoothable.

Theorem 1.1. *Let X be a closed, simply connected, spin topological 4-manifold not homeomorphic to either S^4 or $S^2 \times S^2$. Then, for any sufficiently large prime number p, there exists a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ on X which is nonsmoothable.*

The conclusion of Theorem 1.1 does not hold for S^4. It is known that every pseudofree locally linear action of odd order cyclic group on S^4 is smooth with respect to a smooth structure isomorphic to the standard one (see [15]). Concerning smooth actions on $S^2 \times S^2$, M. Klemm obtained partial results [9], while I do not know whether $S^2 \times S^2$ admits a homologically trivial, pseudofree, nonsmoothable locally linear action or not. The following problem seems open.

Problem 1.2. Is there a homologically trivial, pseudofree, nonsmoothable locally linear action of $\mathbb{Z}_p$ on $S^2 \times S^2$ for some odd prime number p?

Let $NS(X)$ be the set of every prime number p for which X admits a homologically trivial, pseudofree, nonsmoothable locally linear action of $\mathbb{Z}_p$. Theorem 1.1 tells that the complement of $NS(X)$ in the set of prime numbers is bounded for each closed, simply connected, spin 4-manifold X if X is not homeomorphic to S^4 or $S^2 \times S^2$. In the proof of Theorem 1.1, we obtain an estimate of the maximum value of the complement.

Theorem 1.3. *For any closed, simply connected, spin 4-manifold X not homeomorphic to S^4 or $S^2 \times S^2$, $NS(X)$ contains all the prime numbers p satisfying*

$$(1.1) \qquad\qquad p \geq 12 \left[\frac{\max\{b_2^+(X), b_2^-(X)\} + 1}{2} \right] - 5.$$

Here $[x]$ is the maximum integer less than or equal to x. Though we need to fix an orientation of X to define $b_2^+(X)$ and $b_2^-(X)$, the right-hand side of the above estimate of p does not depend on the choice.

The above estimate is not best possible. We show a better estimate for the connected sums of the copies of $S^2 \times S^2$.

Theorem 1.4. *1. $NS(S^2 \times S^2 \# S^2 \times S^2)$ contains all the prime numbers $p \geq 7$.*
2. For $n \geq 3$, $NS(\#^n S^2 \times S^2)$ contains all the prime numbers $p \geq 19$.

We also obtain

Theorem 1.5. $11 \in NS(K3)$.

We prove Theorem 1.1 in three steps. In section 2 we give a family of homologically trivial, pseudofree, locally linear actions, slightly modifying the construction of Edmonds in [5] and making use of the criterion of Edmonds and J. H. Ewing in [6]. In section 3 we calculate the dimension of $\mathbb{Z}_p$-invariant part of the $\mathbb{Z}_p$-index of the Dirac operator for the action constructed in Section 2, assuming that X is spin and that the action is smooth for some smooth structure (Proposition 3.3). The dimension is equal to the index of the Dirac operator on the quotient V-manifold $X/\mathbb{Z}_p$. In section 4 we prove Theorem 1.1 applying the 10/8-type inequality for the quotient V-manifold $X/\mathbb{Z}_p$ in [7].

Remark 1.6. Presumably the estimate in Theorem 1.4 could be improved further in general. We also do not know the set $NS(K3)$ exactly while Theorem 1.3 tells that $NS(K3)$ contains all the prime numbers greater than 113.

Remark 1.7. When a smooth structure is endowed on a topological manifold, a locally linear group action on the topological manifold is called nonsmoothable if the action is not smooth with respect to any smooth structure isomorphic to the given one. W. Chen and S. Kwasik constructed group actions on $K3$ surface of this type, which are smooth with respect to the standard smooth structure but not smooth with respect to infinitely many exotic structures [2]. X. Liu and N. Nakamura constructed group actions on elliptic surfaces which are not smooth with respect to infinitely many smooth structures including the standard smooth structure [12, 13]. It is not known whether the examples of Liu and Nakamura are nonsmoothable for every smooth structure or not. Liu and Nakamura used mod-p vanishing theorem of Seibert-Witten invariant for 4-manifolds with non-vanishing Seiberg-Witten invariant. Recently Nakamura applied a similar method to $K3 \# K3$, for which the Seiberg-Witten invariant is zero but its cohomotopy refinement does not vanish [14]. We also use Seiberg-Witten theory to investigate nonsmoothability of finite group action. Our approach is to apply an equivariant version of 10/8-inequality to spin 4-manifolds, which does not depend on non-vanishing of Seiberg-Witten invariant.

2. Locally linear actions

Let X be a closed, oriented, simply connected topological 4-manifold not necessarily spin. Edmonds proved the following theorem.

Theorem 2.1 (**Edmonds**, Theorem 6.4 in [5])**.** *For any prime number p not less than 5, there exists a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ on X.*

Edmonds constructed the group action using equivariant surgery on the connected sum of $b_2^+(X)$-copies of $\mathbb{C}P^2$ and $b_2^-(X)$-copies of $\overline{\mathbb{C}P^2}$ for some choice of $\mathbb{Z}_p$-action. Moreover Edmonds and Ewing obtained a necessary and sufficient criterion for realizability of a pair of a fixed point data and a unimodular quadratic form with $\mathbb{Z}_p$-action by a pseudofree locally linear $\mathbb{Z}_p$-action on X [6]. In this section we follow Edmonds's construction with a slight modification to obtain a family of fixed point data satisfying Edmonds and Ewing's criterion. More specifically, we make realizable fixed point data by gathering the fixed point data of pseudofree $\mathbb{Z}_p$-actions on $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and S^4.

We identify $\mathbb{Z}_p$ with the subgroup of $U(1)$, and, for an integer a, let $\mathbb{C}_a$ be the one-dimensional complex representation of $\mathbb{Z}_p$ defined by $z \mapsto g^a z$ for $z \in \mathbb{C}$ and $g \in \mathbb{Z}_p$.

Definition 2.2. Using weights $\alpha = (a_0, a_1, a_2)$, $\alpha' = (a'_0, a'_1, a'_2)$ and $\beta = (b_1, b_2)$ respectively, define $\mathbb{Z}_p$-manifolds $\mathbb{C}P^2_\alpha$, $\overline{\mathbb{C}P^2_{\alpha'}}$ and S^4_β as $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and S^4 with pseudofree $\mathbb{Z}_p$-actions as follows.

(1) Suppose a_0, a_1 and a_2 are integers which are not congruent modulo p each other. Let $\mathbb{C}P^2_\alpha$ denote the quotient space $(\mathbb{C}_{a_0} \oplus \mathbb{C}_{a_1} \oplus \mathbb{C}_{a_2} \setminus \{(0,0,0)\})/\mathbb{C}^*$.
(2) Suppose a'_0, a'_1 and a'_2 are integers which are not congruent modulo p each other. Let $\overline{\mathbb{C}P^2_{\alpha'}}$ denote the same as $\mathbb{C}P^2_{\alpha'}$ but with the opposite orientation.
(3) Suppose b_1 and b_2 are integers not congruent to 0 modulo p either. Let S^4_β denote the unit sphere of $\mathbb{C}_{b_1} \oplus \mathbb{C}_{b_2} \oplus \mathbb{R}$, where $\mathbb{R}$ is the trivial one-dimensional real representation of $\mathbb{Z}_p$.

Note that the two weights $\alpha_0 = (a_0, a_1, a_2)$ and $\alpha_1 = (a_0+1, a_1+1, a_2+1)$ give the same action on $\mathbb{C}P^2$, hence $\mathbb{C}P^2_{\alpha_0}$ and $\mathbb{C}P^2_{\alpha_1}$ are $\mathbb{Z}_p$-equivariantly diffeomorphic. From now on we assume that $a_0 + a_1 + a_2$ and $a'_0 + a'_1 + a'_2$ are even for weights $\alpha = (a_0, a_1, a_2)$ and $\alpha' = (a'_0, a'_1, a'_2)$ in Definition 2.2 without loss of generality.

To make a realizable fixed point data by gathering those of $\mathbb{C}P^2_\alpha$'s, $\overline{\mathbb{C}P^2_{\alpha'}}$'s and S^4_β's we may need to reduce the number of fixed points.

Definition 2.3. We call a pair of fixed points *a cancelling pair* if there is a weight β such that the fixed point data of S^4_β coincides with that of the pair. We also call such a weight β a *weight of the cancelling pair*.

A pair of fixed points is a cancelling pair if and only if the two isotropy representations at the two fixed points are isomorphic to each other through an orientation-reversing isomorphism. The weight of the cancelling pair is one of the weights of these representations. (We have two possible representatives of weights for each cancelling pair.)

We will use the following cancelling pairs later. These examples are special cases of Lemma 6.2 in [5].

Example 2.4. Let p be a prime number not less than 5.

(1) Let a, b and c be integers satisfying $a - b, a - b - c, a - b - 2c, c \not\equiv 0 \bmod p$ and $a \equiv c \bmod 2$. For $\alpha_1 = (a, b, b + c)$ and $\alpha_2 = (a, b + c, b + 2c)$, the pair $[0, 0, 1]$ on $\mathbb{C}P^2_{\alpha_1}$ and $[0, 1, 0]$ on $\mathbb{C}P^2_{\alpha_2}$ is a cancelling pair.

(2) Let i be an integer satisfying $i \not\equiv -1, -2, -3 \bmod p$. For $\alpha_1 = (-1, i, i + 1)$ and $\alpha_2 = (-1, i + 1, i + 2)$, the pair of $[0, 0, 1]$ on $\mathbb{C}P^2_{\alpha_1}$ and $[0, 1, 0]$ on $\mathbb{C}P^2_{\alpha_2}$ is a cancelling pair.

(3) For $\alpha_1 = (-1, p - 4, p - 3)$ and $\alpha_2 = (-1, 0, 1)$, the pair of $[0, 0, 1]$ on $\mathbb{C}P^2_{\alpha_1}$ and $[1, 0, 0]$ on $\mathbb{C}P^2_{\alpha_2}$ is a cancelling pair.

(4) Let n be a positive integer. For each $1 \leq i \leq n$ we write $R(i)$ for the remainder of i divided by $p - 3$, and α_i for the weight $(-1, R(i), R(i) + 1)$. There are $n - 1$ cancelling pairs in the fixed points of the disjoint union $\coprod_{1 \leq i \leq n} \mathbb{C}P^2_{\alpha_i}$.

Note that (2) is the special case of (1) with $a = -1$, $b = i$, and $c = 1$, (3) is essentially the case (2) with $i = p - 4$ since weights $(-1, 0, 1)$ and $(p - 3, p - 2, p - 1)$ induce the same action, and (4) is a consequence of the cases (2) and (3).

Proposition 2.5. *Let X be a closed, oriented, simply connected topological 4-manifold, and m, m', r and s non-negative integers satisfying*

$$m - m' = \sigma(X) \quad and \quad 3(m + m') + 2(r + s) = \chi(X),$$

where $\sigma(X)$ and $\chi(X)$ are the signature and the Euler number of X respectively. Suppose there are weights α_i $(1 \leq i \leq m)$, α'_j $(1 \leq j \leq m')$, and β_k $(1 \leq k \leq r)$ such that the fixed point set of the disjoint union

$$\left(\coprod_{1 \leq i \leq m} \mathbb{C}P^2_{\alpha_i} \right) \coprod \left(\coprod_{1 \leq j \leq m'} \overline{\mathbb{C}P^2_{\alpha'_j}} \right) \coprod \left(\coprod_{1 \leq k \leq r} S^4_{\beta_k} \right)$$

has s cancelling pairs. Let $\mathcal{D}$ be the fixed point data for those fixed points which does not appear in the s cancelling pairs. Then there exists a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ on X whose fixed point data is the same as $\mathcal{D}$.

Proof. We check that the data $\mathcal{D}$ satisfies the three conditions REP, GSF, and TOR in [6]. We write Y for the disjoint union in the statement of the theorem. Let γ_l $(1 \leq l \leq s)$ be weights of the s cancelling pairs on Y, and we write Z for the disjoint union

$$Z = \coprod_{1 \leq l \leq s} S^4_{\gamma_l}.$$

Since the number of fixed points of $\mathcal{D}$ is $3m + 3m' + 2r - 2s = \chi(X)$, $\mathcal{D}$ satisfies the condition REP for homologically trivial action on X.

The right-hand side of GSF for $\mathcal{D}$ is the difference between those for the fixed point data of Y and of Z. This is equal to $\sigma(Y) - \sigma(Z)$ since the condition GSF is true for both Y and Z. By the assumption of the proposition, $\sigma(Y) - \sigma(Z) = m - m' = \sigma(X)$. So $\mathcal{D}$ satisfies also the condition GSF for homologically trivial action on X.

The condition TOR is equivalent to the equation of Application 8.6 in [6] for homologically trivial action. Let $L(\mathcal{D})$, $L(Y)$, and $L(Z)$ be the left-hand sides of the equations for $\mathcal{D}$, for the fixed point data of Y, and for that of Z respectively.

Note that we have $L(\mathcal{D}) = L(Y)/L(Z)$. Since the fixed point data of Y and that of Z satisfy

$$L(Y) \approx (-1)^m \times (-(-1))^{m'} \times (-1)^r = (-1)^{m+r} \quad \text{and} \quad L(Z) \approx (-1)^s,$$

we obtain

$$L(\mathcal{D}) \approx (-1)^{m+r-s} = (-1)^{b_2^-(X)+1},$$

which is the equation for $\mathcal{D}$. $\qquad\square$

Edmonds used the construction with $(m, m', r, s) = (b_2^+(X), b_2^-(X), 0, b_2(X)-1)$ to prove Theorem 2.1 in [5]. We will use other choices of (m, m', r, s) to prove Theorems 1.1, 1.4 and 1.5.

3. Index of Dirac operator

In this section, we calculate the dimension of $\mathbb{Z}_p$-invariant part of the $\mathbb{Z}_p$-index of the Dirac operator on X for the $\mathbb{Z}_p$-action given in Proposition 2.5, assuming that X is a spin smooth manifold and that the $\mathbb{Z}_p$-action is smooth.

Recall that we are assuming that $a_0 + a_1 + a_2$ is even for a weight $\alpha = (a_0, a_1, a_2)$. Let $|\alpha|$ be $a_0 + a_1 + a_2$.

Definition 3.1. Define a non-negative integer $N(p, \alpha)$ as the number of ordered triplets of integer (n_0, n_1, n_2) satisfying

$$n_0, n_1, n_2 \geq 0, \quad n_0 + n_1 + n_2 = \frac{p-3}{2}, \quad \text{and} \quad a_0 n_0 + a_1 n_1 + a_2 n_2 + \frac{|\alpha|}{2} \equiv 0 \pmod p.$$

Lemma 3.2. *The dimension of $\mathbb{Z}_p$-invariant part of the $\mathbb{Z}_p$-index of the Dolbeault operator on $\mathbb{C}P_\alpha^2$ with coefficient $\mathcal{O}(\frac{p-3}{2}) \otimes \mathbb{C}_{-\frac{|\alpha|}{2}}$ is equal to $N(p, \alpha)$.*

Proof. Let L_α be the $\mathbb{Z}_p$-equivariant line bundle $\mathcal{O}(\frac{p-3}{2}) \otimes \mathbb{C}_{-\frac{|\alpha|}{2}}$. The $\mathbb{Z}_p$-index of the twisted Dolbeault operator is equal to

$$H^0(\mathbb{C}P^2; \mathcal{O}(L_\alpha)) = \text{Span}\left\{ z_0^{n_0} z_1^{n_1} z_2^{n_2} \,\middle|\, n_0, n_1, n_2 \geq 0, \ n_0 + n_1 + n_2 = \frac{p-3}{2} \right\}$$

with $\mathbb{Z}_p$-action

$$g[z_0, z_1, z_2; z_0^{n_0} z_1^{n_1} z_2^{n_2}] = [g^{a_0} z_0, g^{a_1} z_1, g^{a_2} z_2; g^{-\frac{|\alpha|}{2}} z_0^{n_0} z_1^{n_1} z_2^{n_2}]$$
$$= [g^{a_0} z_0, g^{a_1} z_1, g^{a_2} z_2; g^{-\sum a_i n_i - \frac{|\alpha|}{2}} (g^{a_0} z_0)^{n_0} (g^{a_1} z_1)^{n_1} (g^{n_2} z_2)^{n_2}].$$

This completes the proof. $\qquad\square$

Proposition 3.3. *Let X be a closed, simply connected, spin smooth 4-manifold. Suppose a $\mathbb{Z}_p$-action constructed in Proposition 2.5 is smooth with respect to some smooth structure of X. Then the action has the unique lift to the spin structure on X and the dimension of $\mathbb{Z}_p$-invariant part of the $\mathbb{Z}_p$-index of the Dirac operator on X is equal to*

$$\sum_{i=1}^m N(p, \alpha_i) - \sum_{j=1}^{m'} N(p, \alpha_j') - \frac{\sigma(X)}{8} p.$$

Proof. Let Y and Z be as in the Proof of Theorem 2.5, and X' the disjoint union of X and Z. Note that Z and X' are spin. In general, since p is odd, $\mathbb{Z}_p$-actions on spin manifolds have unique lift to spin structures. Let D_X, D_Z and $D_{X'}$ be the Dirac operators on X, Z and X' respectively. Since $\mathrm{ind}_{\mathbb{Z}_p} D_Z = 0$ we have $\mathrm{ind}_{\mathbb{Z}_p} D_{X'} = \mathrm{ind}_{\mathbb{Z}_p} D_X$. We construct below a $\mathbb{Z}_p$-equivariant spin^c-structure on Y so that it is $\mathbb{Z}_p$-equivariantly spin on a neighborhood of the fixed point set, and compare $\mathrm{ind}_{\mathbb{Z}_p} D_{X'}$ with the $\mathbb{Z}_p$-index of the spin^c-Dirac operator of it.

The $\mathbb{Z}_p$-equivariant spin^c-structure on each $\mathbb{C}P^2_{\alpha_i}$-component of Y is defined so that its spin^c-Dirac operator is identified with the Dolbeault operator twisted by L_{α_i} in Lemma 3.2. If we write K for the canonical line bundle of $\mathbb{C}P^2$, then the square of L_{α_i} is $\mathbb{Z}_p$-equivariantly isomorphic to $K \otimes \mathcal{O}(p)$. Since $\mathcal{O}(p)$ is $\mathbb{Z}_p$-equivariantly trivial on a neighborhood of the fixed point set, L_α is a $\mathbb{Z}_p$-equivariant square root of K there. It implies that the spin^c-structure is $\mathbb{Z}_p$-equivariantly spin on a neighborhood of the fixed point set. The $\mathbb{Z}_p$-equivariant spin^c-structure on each $\overline{\mathbb{C}P^2_{\alpha'_j}}$-component of Y is defined so that the spin^c-Dirac operator is identified with the same twisted Dolbeault operator with opposite parity of degree. The $\mathbb{Z}_p$-equivariant spin^c-structure on each $S^4_{\beta_k}$-component of Y is defined as $\mathbb{Z}_p$-equivariant spin structure.

Let D_Y be the spin^c-Dirac operator on the spin^c-structure defined as above. Since the spin action on X' is isomorphic to the spin^c action on Y on neighborhoods of their fixed point sets, $\mathrm{ind}_g D_{X'}$ and $\mathrm{ind}_g D_Y$ coincide for $g \neq 1 \in \mathbb{Z}_p$. This is a consequence of the localization of the equivariant indices as elements of some localization of equivariant K-groups of the neighborhoods of the fixed point sets, or one could also see it from the Atiyah-Segal-Lefschetz formula. Hence we first obtain

$$
\begin{aligned}
\dim(\mathrm{ind}_{\mathbb{Z}_p} D_{X'})^{\mathbb{Z}_p} - \dim(\mathrm{ind}_{\mathbb{Z}_p} D_Y)^{\mathbb{Z}_p} &= \frac{1}{p} \sum_{g \in \mathbb{Z}_p} \mathrm{ind}_g D_{X'} - \frac{1}{p} \sum_{g \in \mathbb{Z}_p} \mathrm{ind}_g D_Y \\
&= \frac{1}{p} \left(\mathrm{ind}_1 D_{X'} - \mathrm{ind}_1 D_Y \right) \\
&= \frac{1}{p} \left(\mathrm{ind} D_{X'} - \mathrm{ind} D_Y \right).
\end{aligned}
$$

(3.1)

Secondly, from the Hirzebruch signature theorem and a direct calculation, we have

$$
\begin{cases}
\mathrm{ind} D_{X'} &= -\dfrac{\sigma(X')}{8} = -\dfrac{\sigma(X)}{8}, \\
\mathrm{ind} D_Y &= (m - m') \dim H^0 \left(\mathbb{C}P^2; \mathcal{O}(\tfrac{p-3}{2}) \right) = \sigma(X) \dfrac{p^2 - 1}{8},
\end{cases}
$$

which implies

(3.2) $$ \frac{1}{p} \left(\mathrm{ind} D_{X'} - \mathrm{ind} D_Y \right) = -\sigma(X) \frac{p}{8}, $$

Thirdly, applying Lemma 3.2 to each component of Y, we have

(3.3) $$ \dim(\mathrm{ind}_{\mathbb{Z}_p} D_Y)^{\mathbb{Z}_p} = \sum_{i=1}^{m} N(p, \alpha_i) - \sum_{j=1}^{m'} N(p, \alpha'_j). $$

Now the equations(3.1), (3.2) and (3.3) imply the required formula. $\qquad\square$

Remark 3.4. The dimension of $\mathbb{Z}_p$-invariant part of the $\mathbb{Z}_p$-index of the Dirac operator on X is nothing but the index of the Dirac operator on the quotient spin V-manifold $X/\mathbb{Z}_p$.

4. Nonsmoothability

In this section, we prove Theorem 1.1 choosing appropriate weights and using *the 10/8-type inequality for the quotient V-manifold $X/\mathbb{Z}_p$ in [7].*

Theorem 4.1. *Let X be a closed, simply connected, spin smooth 4-manifold not homeomorphic to S^4. Suppose the integers m, m', r, s and $\mathbb{Z}_p$-manifolds $\mathbb{C}P^2_{\alpha_i}$ $(1 \le i \le m)$, $\overline{\mathbb{C}P}^2_{\alpha'_j}$ $(1 \le j \le m')$, $S^4_{\beta_k}$ $(1 \le k \le r)$ satisfy the assumption of Proposition 2.5. If the $\mathbb{Z}_p$-action on X constructed in Proposition 2.5 is smooth with respect to some smooth structure of X, then the inequality*

$$-b_2^-(X) < \sum_{i=1}^{m} N(p, \alpha_i) - \sum_{j=1}^{m'} N(p, \alpha'_j) - \frac{\sigma(X)}{8}p < b_2^+(X)$$

holds.

Proof. In general when a finite group G acts on a closed, spin smooth 4-manifold W preserving its orientation and the spin structure, Y. Fukumoto and M. Furuta [7] showed the inequality

$$\dim(\mathrm{ind}_G D)^G < \dim_{\mathbb{R}} H_+^2(W; \mathbb{R})^G$$

when the right-hand side is not zero, where D is the G-equivariant Dirac operator on W. In our case, since the action is homologically trivial, the right-hand side for $W = X$ is equal to $b_2^+(X)$. When X is a spin smooth manifold not homeomorphic to S^4, then a theorem of S. K. Donaldson [3] implies $b_2^+(X), b_2^-(X) > 0$. Therefore if the action is smoothable, we have the above inequality. Using the formula given by Proposition 3.3, we can write the inequality as

$$\sum_{i=1}^{m} N(p, \alpha_i) - \sum_{j=1}^{m'} N(p, \alpha'_j) - \frac{\sigma(X)}{8}p < b_2^+(X).$$

Reversing the orientation of X, we similarly obtain

$$\sum_{j=1}^{m'} N(p, \alpha'_j) - \sum_{i=1}^{m} N(p, \alpha_i) + \frac{\sigma(X)}{8}p < b_2^-(X).$$

$\square$

Proof of Theorems 1.1 and 1.3. When X has no smooth structure Theorem 1.1 is included in Edmonds's Theorem 2.1. So we assume below that X has a smooth structure. We also $\sigma(X) \le 0$ giving the opposite orientation to X if necessary.

To construct an action which does not satisfy the inequality of Theorem 4.1 we choose different triplet (m, m', r, s) from that used by Edmonds in [5].

Lemma 4.2. *Let p be a prime number not less than 5, and X a closed, simply connected, spin smooth 4-manifold with $\sigma(X) \le 0$ not homeomorphic to S^4 or $S^2 \times S^2$. Then there exist weights*

$$\alpha'_1, \alpha'_2, \ldots, \alpha'_{-\sigma(X)}$$

for $\overline{\mathbb{CP}^2}$ satisfying the following property: For any weights α_0 and α_0', the $\mathbb{Z}_p$-manifolds $\mathbb{CP}_{\alpha_0}^2$, $\overline{\mathbb{CP}^2}_{\alpha_j'}$ $(0 \leq j \leq -\sigma(X))$ and $S_{\beta_k}^4$ $(1 \leq k \leq r)$ for some non-negative integer r and some weights β_k $(1 \leq k \leq r)$ satisfy the assumption of Proposition 2.5.

Proof. For each $1 \leq j \leq -\sigma(X)$ we write $R(j)$ for the remainder of j divided by $p-3$. We show that the weights $\alpha_j' = (-1, R(j), R(j)+1)$ $(1 \leq j \leq -\sigma(X))$ satisfy the required property. Recall that, in the assumption of Proposition 12, m, m', r and s are non-negative integers satisfying $m - m' = \sigma(X)$ and $3(m+m')+2(r+s) = \chi(X)$.

Case I: If $-3\sigma(X) + 6 \leq \chi(X)$, then we take

$$m = 1, \quad m' = -\sigma(X) + 1, \quad r = \frac{3\sigma(X) - 6 + \chi(X)}{2}, \quad \text{and} \quad s = 0.$$

Since we do not require existence of cancelling pairs, any choice of $\mathbb{Z}_p$-manifolds satisfies the assumption of Proposition 2.5.

Case II: If $-3\sigma(X) + 6 > \chi(X)$, then we take

$$m = 1, \quad m' = -\sigma(X) + 1, \quad r = 0, \quad \text{and} \quad s = \frac{-3\sigma(X) + 6 - \chi(X)}{2}.$$

Our assumption implies $s \geq 0$. We will show the inequality $-\sigma(X) - 1 \geq s(\geq 0)$. Then the proof will be completed because Example 2.4 (4) tells that, under the inequality $-\sigma(X) - 1 \geq 0$, the number of the cancelling pairs is at least $-\sigma(X) - 1$, and hence at least s under the inequality $s \leq -\sigma(X) - 1$. Since $-\sigma(X) - 1 - s = b_2^+(X) - 3$, it suffices to show $b_2^+(X) \geq 3$. If not, and if X is smooth, Donaldson's Theorems B and C in [4] imply $b_2^+(X) = b_2^-(X) \leq 2$, i.e., $\sigma(X) = 0$ and $\chi(X) = 2$, 4, or 6. The case $\chi(X) = 6$ is excluded from the assumption $-3\sigma(X) + 6 > \chi(X)$ of Case II. The cases $\chi(X) = 2$ and 4 are also excluded from our assumption that X is not homeomorphic to S^4 or $S^2 \times S^2$. $\qquad\square$

We continue to prove Theorems 1.1 and 1.3.

Fix a prime number p not less than 5. We consider the actions constructed in the proof of Lemma 4.2. So we use the notation there. Choose and fix weights α_j' for $1 \leq j \leq -\sigma(X)$ as in Lemma 4.2 so that the union of fixed points of the $\mathbb{Z}_p$-manifolds $\overline{\mathbb{CP}_{\alpha_j'}^2}$ for $1 \leq j \leq -\sigma(X)$ has s cancelling pairs. We can choose arbitrarily the rest of weights α_0, α_0', and β_k for $1 \leq k \leq r$. We would like to choose these weights so that the inequality of Theorem 4.1 is violated. Since the fixed point data of S_β^4 does not contribute to the $\mathbb{Z}_p$-index of the Dirac operator on X, what we can effectively control are the two weights α_0 and α_0'.

If we write I for the integer

$$I = - \sum_{j=1}^{-\sigma(X)} N(p, \alpha_j') - \frac{\sigma(X)}{8}p,$$

then Proposition 3.3 implies

$$\dim(\mathrm{ind}_{\mathbb{Z}_p} D_X)^{\mathbb{Z}_p} = I + N(p, \alpha_0) - N(p, \alpha_0').$$

On the other hand, a direct calculation shows that

$$N(p, (-1, 0, 1)) \;\; = \;\; k \qquad \text{for} \quad p = 4k \pm 1,$$

$$N(p, (-1, 1, 2)) \;\; = \;\; \begin{cases} l - 1 & \text{for} & p = 12l - 5 \\ l & \text{for} & p = 12l \pm 1 \\ l + 1 & \text{for} & p = 12l + 5 \end{cases} \;\; ,$$

which implies

$$(4.1) \quad N(p, (-1, 0, 1)) - N(p, (-1, 1, 2)) = 2l \qquad \text{for} \quad p = 12l + q \;\; (q = \pm 1, \pm 5).$$

In particular, if we choose $\alpha_0 = (-1, 0, 1)$ and $\alpha_0' = (-1, 1, 2)$, then we have

$$\dim(\text{ind}_{\mathbb{Z}_p} D_X)^{\mathbb{Z}_p} = I + 2l \quad \text{for} \quad p = 12l + q \;\; (q = \pm 1, \pm 5),$$

and if we choose $\alpha_0 = (-1, 1, 2)$ and $\alpha_0' = (-1, 0, 1)$ then we have

$$\dim(\text{ind}_{\mathbb{Z}_p} D_X)^{\mathbb{Z}_p} = I - 2l \quad \text{for} \quad p = 12l + q \;\; (q = \pm 1, \pm 5).$$

Therefore at least one of the absolute values of the above two is greater than or equal to $2l$. Hence if p is large enough to satisfy $2l \geq \max\{b_2^+(X), b_2^-(X)\}$, or the inequality (1.1), then one of the above actions does not satisfy the inequality given in Theorem 4.1. This implies the action is nonsmoothable. $\qquad\square$

Remark 4.3. 1. Our construction is not available to find a nonsmoothable $\mathbb{Z}_5$-action, even if it exists. It is because $N(5, \alpha) = 1$ for any weight α, and hence $\dim(\text{ind}_{\mathbb{Z}_5} D_X)^{\mathbb{Z}_5} = 3\sigma(X)/5$ for any $\mathbb{Z}_5$-action on any spin 4-manifold constructed in Proposition 2.5. This value $3\sigma(X)/5$ satisfies the inequality of Theorem 4.1.

2. Our construction is not available to find a nonsmoothable $\mathbb{Z}_p$-action on $S^2 \times S^2$, even if it exists. It is because the fixed point data of any action on $S^2 \times S^2$ constructed in Proposition 2.5 is realized by a smooth action (Lemma 5.1 in [16]).

5. Estimate of p

In the proof of Theorems 1.1 and 1.3 in the previous section, we made use of particular choices of weights. If we use other choices of weights, it is likely that we could construct nonsmoothable actions for some other prime numbers as well. Theorems 1.4 and 1.5 are examples of this kind.

Proof of Theorem 1.4. We take m, m', r and s satisfying $n+1 = 3m+r$, $0 \leq r \leq 2$, $m' = m$ and $s = 0$. Any choice of $\mathbb{C}P_{\alpha_i}^2$ $(1 \leq i \leq m)$, $\overline{\mathbb{C}P_{\alpha_j'}^2}$ $(1 \leq j \leq m)$, and $S_{\beta_k}^4$ $(1 \leq k \leq r)$ satisfies the assumption of Proposition 2.5.

Take $\alpha_i = (-1, 0, 1)$ for every $1 \leq i \leq m$, and $\alpha_j' = (-1, 1, 2)$ for every $1 \leq j \leq m$. If this action is smooth with respect to some smooth structure on X then

$$\dim(\text{ind}_{\mathbb{Z}_p} D_X)^{\mathbb{Z}_p} = mN(p, (-1, 0, 1)) - mN(p, (-1, 1, 2))$$
$$= 2lm \quad \text{for} \quad p = 12l + q \;\; (q = \pm 1, \pm 5)$$

by Proposition 3.3 and the equation (4.1). Hence, for any n and p satisfying

$$2lm \geq n = 3m + r - 1 \quad \text{for} \quad p = 12l + q \;\; (q = \pm 1, \pm 5),$$

the action does not satisfy the inequality of Theorem 4.1. This implies Theorem 1.4. $\qquad\square$

Proof of Theorem 1.5. Let p be 11 and X the topological manifold homeomorphic to $K3$ surface.

For $\alpha'_1 = (-1, 1, 2)$, $\alpha'_2 = (-1, 2, 3)$, and $\alpha'_3 = (-1, 3, 4)$, the three $\mathbb{Z}_p$-manifolds $\overline{\mathbb{C}P^2_{\alpha'_1}}$, $\overline{\mathbb{C}P^2_{\alpha'_2}}$, and $\overline{\mathbb{C}P^2_{\alpha'_3}}$ have two cancelling pairs, as in Example 2.4 (2). On the other hand, for $\alpha'_4 = (-2, 2, 4)$, the pair consisting of $[1, 0, 0]$ on $\overline{\mathbb{C}P^2_{\alpha'_3}}$ and $[1, 0, 0]$ on $\overline{\mathbb{C}P^2_{\alpha'_4}}$ is a cancelling pair. Hence, by taking weights α'_j for $1 \leq j \leq 16$ as

$$\alpha'_j = \alpha'_k \qquad \text{for} \qquad j \equiv k \mod 4 \qquad \text{and} \qquad 1 \leq k \leq 4,$$

16 $\mathbb{Z}_p$-manifolds $\overline{\mathbb{C}P^2_{\alpha'_j}}$ $(1 \leq j \leq 16)$ have 12 cancelling pairs, that is, they satisfy the assumption of Proposition 2.5 for $m = r = 0$, $m' = 16$ and $s = 12$.

Since

$$N(11, (-1, 1.2)) = N(11, (-1, 2, 3)) = N(11, (-1, 3, 4)) = N(11, (-2, 2, 4)) = 1,$$

if the above action is smooth with respect to some smooth structure of X then

$$\dim(\mathrm{ind}_{\mathbb{Z}_{11}} D)^{\mathbb{Z}_{11}} = 6$$

by Lemma 3.3. The action, therefore, does not satisfy the inequality of Theorem 4.1 because $b_2^+(X)$ is equal to 3. This implies the action is nonsmoothable. $\square$

Corollary 5.1. *We have* $11 \in NS(K3 \# (\#^t S^2 \times S^2))$ *for* $t = 1, 2$ *and* 3.

Proof. Let X be $K3 \# (\#^t S^2 \times S^2)$ for $t = 1, 2$, or 3. Choose weights β_k for $1 \leq k \leq t$ arbitrarily, then the $\mathbb{Z}_p$-manifolds $\overline{\mathbb{C}P^2_{\alpha'_j}}$ for $1 \leq j \leq 16$ in the proof of Theorem 1.5 and $S^4_{\beta_k}$ for $1 \leq k \leq t$ satisfy the assumption of Proposition 2.5 for X.

If the action is smoothable,

$$\dim(\mathrm{ind}_{\mathbb{Z}_{11}} D_X)^{\mathbb{Z}_{11}} = 6$$

because the fixed points from $S^4_{\beta_k}$ does not contribute. Then the inequality of Theorem 4.1 is not satisfied, which implies that the action is nonsmoothable. $\square$

Remark 5.2. In the case of $n = 2$ or 3, the estimate of p in Theorem 1.4 coincides with those in Theorem 1.1 and it is not an improvement. In the case of $n \geq 4$, Theorem 1.4 gives an improvement. Still better estimations might be obtained using the construction in Section 2 using other choices of m, m', r and s.

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